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On construction of universal twist element from R-matrix

Abstract

A method to construct the universal twist element using the constant quasiclassical unitary matrix solution of the Yang - Baxter equation is proposed. The method is applied to few known R-matrices, corresponding to Lie (super) algebras of rank one.

1 Introduction

Quantum groups, as an important class of Hopf algebras, were introduced by V.G.Drinfel'd [1]. In the theory of quantum groups the universal R-matrix is an essential object intertwining the coproduct Δ with the opposite coproduct Δ^{op}

$$\mathcal{R}\Delta = \Delta^{\mathrm{op}}\mathcal{R}. \tag{1}$$

Another important object, introduced later, is a twist transformation (or twist) [2, 3]

$$\Delta \to \Delta_t = \mathcal{F}\Delta\mathcal{F}^{-1}, \tag{2}$$

where $\mathcal{R}$ and $\mathcal{F}$ are some elements of the tensor square of the Hopf algebra. Twist elements are known explicitly for some quantum groups, in particular, for quantum deformations of the universal enveloping algebras of some Lie algebras.

The FRT-formalism [4] permits to reconstruct the quantum group corresponding to some known matrix representation of universal R-matrix. However this method does not give explicit expressions for the universal elements.

A similar result for universal twists was obtained by Drinfel'd [2]. It was shown that, if a unitary matrix R being a quasiclassical solution of the Yang - Baxter equation, is known then there exists a universal twist $\mathcal{F}$, such that $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ in the vector representation. This situation is similar to the mentioned above: if a matrix solution of the Yang - Baxter equation is known then a formal reconstruction of the universal element is possible. However, as in the previous case, there is no computational method to obtain explicit expressions for the universal twist $\mathcal{F}$. Constructing of universal twist elements is a rather complicated problem. Some explicit expressions can be found in papers [5, 6] (and refs therein).

In the present paper we suggest a method of constructing of a universal twist element for a given unitary quasiclassical matrix solutions of the Yang - Baxter equation. This method is based on an investigation of the associative algebra defined by the matrix relation $RT_1T_2 = T_2T_1$,

with $T = \{t_{ij} \in U(gl_n)^*\}$. This algebra is a deformation of the algebra of polynomial functions on the group GL_n (eventually, in examples, it is possible to consider its subgroups). The mentioned above associative algebra is isomorphic, as a left $U(g\ell_n)$-module algebra, to the nondeformed (commutative) algebra Fun of polynomials on GL_n (certainly this isomorphism is not an algebra isomorphism). Let us recall that there exists a non-degenerate pairing between Fun and $U(g\ell_n)$. Hence, there is a non-degenerate pairing between $U(g\ell_n)$ and the algebra, generated by the relation $RT_1T_2 = T_2T_1$. This duality generates a coassociative operation $d : U(g\ell_n) \to U(g\ell_n) \otimes U(g\ell_n)$ (which is not an algebra homomorphism). Then it turns out that $\mathcal{F} = d(1)$ is the required twist.

This twist $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ allows to introduce in the Hopf algebra $\mathcal{H}$ an additional coalgebra structure:

$$d : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \quad d := \mathcal{F}\Delta.$$

Structures of this type were considered in papers [7, 8] devoted to analysis of Lie - Poisson groups.

In section 2 we discuss a general connection between the universal twist $\mathcal{F}$ of a Hopf algebra $\mathcal{H}$ and the matrix form of the RTT - relation, which defines a new multiplication law on the dual Hopf algebra $\mathcal{H}^*$. In section 3 we describe some examples of application of this connection for reconstructing of the universal twisting element from a given R-matrix in the case of Lie (super)algebras of rank 1. Several details of calculations are given in the Appendix.

2 Module algebras and twists

Let us consider in a given Hopf algebra $\mathcal{H}$ an element $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$, satisfying the twist equation (the 2-cocycle condition)

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}$$

and relation

$$(\varepsilon \otimes id)\mathcal{F} = (id \otimes \varepsilon)\mathcal{F} = 1 \otimes 1.$$

Such an element $\mathcal{F}$ is called the *universal twist* or the *universal twist element*.

Define a map $d_{\mathcal{F}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, such that $d_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)$. It is easy to show, that the map $d_{\mathcal{F}}$ satisfy the following conditions

$$(d \otimes id)d = (id \otimes d)d \qquad \text{coassociativity}$$

and

$$d_{\mathcal{F}}(ab) = d_{\mathcal{F}}(a)\Delta(b).$$

Lemma 2.1. *For the twist $\mathcal{F}$ the following conditions hold:*

(a) $d_{\mathcal{F}}$ *is a coassociative map.*

(b) $d_{\mathcal{F}}(ab) = d_{\mathcal{F}}(a)\Delta(b)$

(c) $(\varepsilon \otimes id)d_{\mathcal{F}}(a) = 1 \otimes a$ *and* $(id \otimes \varepsilon)d_{\mathcal{F}}(a) = a \otimes 1$.

Conversely, if $d : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ satisfies the conditions (a), (b) *and* (c)*, then $\mathcal{F} = d(1)$ is a twist.*

Proof. We have already shown, that the twist $\mathcal{F}$ generates a map $d_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)$, satisfying the conditions (a), (b), while (c) is straightforward.

Now we prove that if the map $d : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is given, then from the properties (a), (b) it follows, that the map

$$\mathcal{F} = d(1) := \sum f_1 \otimes f_2$$

satisfies the relation

$$\mathcal{F}_{12}(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}(id \otimes \Delta)(\mathcal{F})\,.$$

Then, by putting $a = 1$ in (c) we obtain $(\varepsilon \otimes id)\mathcal{F} = (id \otimes \varepsilon)\mathcal{F} = 1 \otimes 1$. $\square$

Let us consider the vector space $\mathcal{H}^*$ dual to $\mathcal{H}$. We note that $\mathcal{H}^*$ is a right $\mathcal{H}_{op}$-module since

$$\langle \alpha, ab \rangle = \langle \alpha_{(1)}, a \rangle \langle \alpha_{(2)}, b \rangle := \langle \alpha\tilde{b}, a \rangle, \quad \alpha\tilde{b} = \alpha_{(1)} \langle \alpha_{(2)}, b \rangle.$$

where $\alpha \in \mathcal{H}^*$, $a, b \in \mathcal{H}$ and $\mathcal{H}_{op}$ is $\mathcal{H}$ with the opposite multiplication. The map $d : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, satisfying the conditions (a) - (c) induces an associative multiplication law on $\mathcal{H}^*$

$$\begin{gathered}
\langle \alpha \circ \beta, a \rangle = \langle \alpha \otimes \beta, d(a) \rangle = \langle \alpha \otimes \beta, f_1 a_{(1)} \otimes f_2 a_{(2)} \rangle = \\
\langle \alpha, f_1 a_1 \rangle \langle \beta, f_2 a_{(2)} \rangle = \langle \Delta^*(\alpha), f_1 \otimes a_{(1)} \rangle \langle \Delta^*(\beta), f_2 \otimes a_{(2)} \rangle = \\
\langle \alpha_{(1)}, f_1 \rangle \langle \alpha_{(2)}, a_{(1)} \rangle \langle \beta_{(1)}, f_2 \rangle \langle \beta_{(2)}, a_{(2)} \rangle = \\
\langle \alpha_{(1)}, f_1 \rangle \langle \beta_{(1)}, f_2 \rangle \langle \alpha_{(2)}, a_{(1)} \rangle \langle \beta_{(2)}, a_{(2)} \rangle = \\
\langle \alpha_{(1)}, f_1 \rangle \langle \beta_{(1)}, f_2 \rangle \langle \alpha_{(2)} \otimes \beta_{(2)}, a_{(1)} \otimes a_{(2)} \rangle = \\
\langle \alpha_{(1)}, f_1 \rangle \langle \beta_{(1)}, f_2 \rangle \langle \alpha_{(2)} \otimes \beta_{(2)}, \Delta(a) \rangle = \langle \alpha_{(1)} \otimes \beta_{(1)}, \mathcal{F} \rangle \langle \alpha_{(2)} \otimes \beta_{(2)}, \Delta(a) \rangle = \\
\langle \alpha_{(1)} \otimes \beta_{(1)} \otimes \alpha_{(2)} \otimes \beta_{(2)}, \mathcal{F} \otimes \Delta(a) \rangle = \langle \sigma_{23} \left(\alpha_{(1)} \otimes \alpha_{(2)} \otimes \beta_{(1)} \otimes \beta_{(2)} \right), \mathcal{F} \otimes \Delta(a) \rangle = \\
\langle \sigma_{23} \Delta^*(\alpha) \otimes \Delta^*(\beta), \mathcal{F} \otimes \Delta(a) \rangle\,.
\end{gathered}$$

Hence $\alpha \circ \beta = \langle \alpha_{(1)} \otimes \beta_{(1)}, \mathcal{F} \rangle \alpha_{(2)} \beta_{(2)}$.

Proposition 2.2. *Twist elements are in a one-to-one correspondence with module algebra structures on $\mathcal{H}^*$ (over $\mathcal{H}_{op}$), such that $\varepsilon \circ f = f \circ \varepsilon = f$.*

Corollary 2.3. *Let $\{e_0 = 1, e_1, e_2, \ldots\}$ and $\{e^0 = \varepsilon, e^1, e^2, \ldots\}$ be the dual bases in $\mathcal{H}$ and $\mathcal{H}^*$, respectively, and let*

$$e^i \circ e^j = \sum_k m_k^{ij} e^k.$$

Then

$$d(1) = \sum_{r,s} m_0^{rs} e_r \otimes e_s \quad \textit{and} \quad m_r^{0k} = m_r^{k0} = \delta_r^k.$$

Indeed,

$$\begin{aligned}
\left\langle e^j \circ e^i, e_p \right\rangle &= \left\langle e^j \otimes e^i, d(e_p) \right\rangle \\
\left\langle e^j \circ e^i, e_p \right\rangle &= \left\langle \sum\nolimits_k \overset{\circ}{m}{}_k^{j\,i}\; e^k, e_p \right\rangle = \sum\nolimits_k \overset{\circ}{m}{}_k^{j\,i} \left\langle e^k, e_p \right\rangle = \overset{\circ}{m}{}_p^{j\,i} \\
\left\langle e^j \otimes e^i, d(e_p) \right\rangle &= \left\langle e^j \otimes e^i, \sum\nolimits_{r,s} d_p^{r\,s} e_r \otimes e_s \right\rangle = \sum\nolimits_{r,s} d_p^{r\,s} \left\langle e^j \otimes e^i, e_r \otimes e_s \right\rangle = \\
&= \sum\nolimits_{r,s} d_p^{r\,s} \left\langle e^j, e_r \right\rangle \left\langle e^i, e_s \right\rangle = d_p^{j\,i}.
\end{aligned}$$

Thus $d_p^{j\,i} = \overset{\circ}{m}{}_p^{j\,i}$ and

$$\begin{aligned}
d(e_p) &:= \sum\nolimits_{r,s} d_p^{r\,s} e_r \otimes e_s = \sum\nolimits_{r,s} \overset{\circ}{m}{}_p^{r\,s}\; e_r \otimes e_s\,, \\
d(e_p) &= d(1 e_p) = d(1)\Delta(e_p) = \mathcal{F}\Delta(e_p)\,.
\end{aligned}$$

3 Examples

3.1 The case of $U(b_2)$

Let $\mathcal{H} = U(b_2)$ be the universal enveloping algebra of the Borel algebra with generators h, x and the defining relation $[h, x] = x$. It is known, that monomials $e_{m,k} = h^m x^k, m, k = 0, 1, 2, \ldots$ form a linear basis in $\mathcal{H}$ (PBW-theorem). Let us consider the jordanian R-matrix

$$R = \begin{pmatrix} 1 & -\xi & \xi & \xi^2 \\ 0 & 1 & 0 & -\xi \\ 0 & 0 & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is a unitary solution of the Yang - Baxter equation [9] (that is $R_{21}R = 1 \otimes 1$) and quasiclassical. Therefore according to [2] there exists a twist $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ such, that $R = (\rho \otimes \rho)(\mathcal{F}_{21}\mathcal{F}^{-1})$, where ρ is the vector representation of $U(b_2)$. To find $\mathcal{F}$, we shall consider the following algebra, (it is not a Hopf algebra):

$$\mathcal{A} = \frac{\mathbb{C}\langle t_{11}, t_{12} \rangle}{R T_1 \cdot T_2 = T_2 \cdot T_1}$$

Here

$$T_1 = T \otimes 1 = \begin{pmatrix} t_{11} & 0 & t_{12} & 0 \\ 0 & t_{11} & 0 & t_{12} \\ 0 & 0 & t_{11}^{-1} & 0 \\ 0 & 0 & 0 & t_{11}^{-1} \end{pmatrix}$$

and

$$T_2 = 1 \otimes T = \begin{pmatrix} t_{11} & t_{12} & 0 & 0 \\ 0 & t_{11^{-1}} & 0 & 0 \\ 0 & 0 & t_{11} & t_{12} \\ 0 & 0 & 0 & t_{11}^{-1} \end{pmatrix}$$

Let $\mathbb{C}\langle t_{11}, t_{12}\rangle$ be an algebra, generated by the noncommutative elements t_{11}, t_{12} satisfying the relation $RT_1 \cdot T_2 = T_2 \cdot T_1$.

Proposition 3.1. 1) $\mathcal{A} = \dfrac{\mathbb{C}\langle t_{11}, t_{12}\rangle}{[t_{11}, t_{12}] = \xi \cdot 1}$

2) *$\mathcal{A}$ is a left module algebra over $\mathcal{H} = U(b_2)$ with the action:*

$$\tilde{x}t_{12} = t_{11},\ \tilde{h}t_{12} = -\frac{1}{2}t_{12},\ \tilde{x}t_{11} = 0,\ \tilde{h}t_{11} = \frac{1}{2}t_{11}.$$

Proof. Both statements can be checked immediately. It is also possible to prove these statements as follows. First of all, it is easy to prove the Poincare - Birkhoff - Witt theorem: the monomials

$$f^{k,m} = t_{11}^k t_{12}^n$$

form a linear basis in $\mathcal{A}$. Let us consider the algebra of polynomials on the group B_2

$$Fun = \frac{\mathbb{C}\langle t_{11}, t_{12}\rangle}{T_1T_2 = T_2T_1} = \mathbb{C}[t_{11}, t_{12}]\,.$$

It is known, that Fun is both a left $U(b_2)$ - module algebra and a right $U(b_2)_{op}$ - module algebra. It is also known that $U(b_2)$ acts on Fun by left derivatives whereas $U(b_2)_{op}$ acts by right derivatives. Because of in the defining relation R stands on the left, it is possible to show, that the structure of the left $U(b_2)$ - module algebra on $\mathcal{A}$ is inherited from Fun. □

Corollary 3.2. *As left $U(b_2)$ - module algebras $\mathcal{A}$ and Fun are isomorphic.*

Let us recall, that the used pairing between $U(b_2)$ and Fun is determined by the relation $\langle f, b\rangle = f\widetilde{b}(e)$, in which $f \in Fun, b \in U(b_2)$, and e is the unit element of the group B_2.

So, we have to find dual bases in $U(b_2)$ and Fun. However, it is known, that it is not possible therefore we expand algebra Fun. We will construct such an extension according to the method suggested in [10]. We recall, that Fun is the Hopf algebra, and the mentioned above dualization is the pairing of the Hopf algebras Fun and $U(b_2)_{op}$.

Let φ and ω denote the elements dual to h and x, respectively. Then

$$\langle h^k x^m, \varphi^p \omega^q\rangle = k!\, m! \cdot \delta_{k,p} \cdot \delta_{m,q},$$

and the tensor

$$\mathcal{T} = \exp(h \otimes \varphi)\exp(x \otimes \omega) \in U(b_2) \otimes \widetilde{Fun}$$

is the canonical element. Here we have denoted the Hopf algebra, generated by the elements $\varphi, \omega, e^{\pm\varphi/2}$ by $\widetilde{Fun}$ with $\Delta\varphi$, $\Delta\omega$ defined by

$$\langle \Delta(a), h^k x^m \otimes h^p x^q\rangle = \langle a, h^k x^m h^p x^q\rangle.$$

It is possible to check, that φ is a primitive element, and $\Delta\omega = \omega \otimes e^{-\varphi} + 1 \otimes \omega$. Thus, it is clear that $\widetilde{Fun}$ is a right module algebra over $U(b_2)_{op}$.

If $\mathcal{F} \in U(b_2) \otimes U(b_2)$ is a twist, then we will define a new multiplication in $\widetilde{Fun}$ as follows: since $\mathcal{F}$ defines a coassociative operation $d_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)$ on $U(b_2)$, then by virtue of the duality between $U(b_2)$ and $\widetilde{Fun}$ we obtain the following associative multiplication on $\widetilde{Fun}$, which can be described as follows.

Let us define

$$\mathcal{T}_1 = \exp(h \otimes 1 \otimes \varphi) \exp(x \otimes 1 \otimes \omega) \quad \text{and} \quad \mathcal{T}_2 = 1 \otimes \mathcal{T},$$

and also

$$\widetilde{T}_1 = (\rho \otimes \rho \otimes id)\mathcal{T}_1 \quad \text{and} \quad \widetilde{T}_2 = (\rho \otimes \rho \otimes id)\mathcal{T}_2,$$

where ρ denotes 2-dimensional vector representation of the algebra $U(b_2)$. Then $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1$. Further we will define $\mathcal{T}_1 \circ \mathcal{T}_2$ by

$$\mathcal{T}_1 \circ \mathcal{T}_2 = \mathcal{F}_{12}(\Delta \otimes id)(\mathcal{T}).$$

Then it is possible to prove, that

$$\mathcal{T}_2 \circ \mathcal{T}_1 = \mathcal{F}_{21}(\Delta_{op} \otimes id)(\mathcal{T})$$

Indeed, if we represent $\mathcal{T}$ as $\sum e_i \otimes e^i$, with dual bases $\{e_i\}, \{e^i\}$ in $U(b_2)$ and $\widetilde{Fun}$, respectively, we obtain

$$\mathcal{T}_1 \circ \mathcal{T}_2 = \sum e_i \otimes e_j \otimes e^i \circ e^j = \mathcal{F}_{12}(\Delta \otimes id)(\mathcal{T}).$$

Calculating $\mathcal{T}_2 \circ \mathcal{T}_1$, we get

$$\mathcal{T}_2 \circ \mathcal{T}_1 = \sum e_i \otimes e_j \otimes e^j \circ e^i = \mathcal{F}_{21}(\Delta_{op} \otimes id)(\mathcal{T}).$$

Let $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$ and $R = (\rho \otimes \rho)(\mathcal{R})$. Due to the equality $\Delta = \Delta_{op}$, we have proved

Theorem 3.3. *$\mathcal{R}\mathcal{T}_1 \circ \mathcal{T}_2 = \mathcal{T}_2 \circ \mathcal{T}_1$ and, consequently, $R\tilde{T}_1 \circ \tilde{T}_2 = \tilde{T}_2 \circ \tilde{T}_1$.*

This theorem allows us to construct a twist for the given jordanian R-matrix (and not only for the jordanian one).

Taking into account, that

$$h = \frac{1}{2}\sigma^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

after simple calculations we obtain

$$\widetilde{T} = \begin{pmatrix} e^{\varphi/2} & e^{\varphi/2} \cdot \omega \\ 0 & e^{-\varphi/2} \end{pmatrix}.$$

Then it is possible to rewrite the defining relation $R\widetilde{T}_1 \circ \widetilde{T}_2 = \widetilde{T}_2 \circ \widetilde{T}_1$ as follows

$$e^{\varphi/2} \circ \omega = \omega \circ e^{\varphi/2} + \xi e^{-\varphi/2}.$$

Remark 3.4. *Let us recall, that*

$$\mathcal{A} = \frac{\mathbb{C}\langle t_{11}, t_{12}\rangle}{RT_1 \cdot T_2 = T_2 \cdot T_1}$$

is a right module algebra over $\mathcal{H}_{op} = U(b_2)_{op}$. Standard considerations (note, that we should start from the algebra $\widetilde{Fun}$, which is a left $\mathcal{H}$-module and a right $\mathcal{H}_{op}$-module) show that the algebra, generated by $\widetilde{T}$ with the defining relation $R\widetilde{T}_1 \circ \widetilde{T}_2 = \widetilde{T}_2 \circ \widetilde{T}_1$ is a right $\mathcal{H}_{op}$-module isomorphic to $\widetilde{Fun}$ as a right $\mathcal{H}_{op}$-module. Advantage of using $\widetilde{T}$ instead of T is, the existence of the dual bases in this case. The isomorphism is given by the correspondence:

$$\varphi^p \omega^q \longrightarrow \varphi \circ \varphi \ldots \circ \varphi \circ \omega \circ \ldots \circ \omega\,,$$

where we use a short-hand notation $\varphi^p \circ \omega^q$.

Let us rewrite the relation

$$e^{\varphi/2} \circ \omega = \omega \circ e^{\varphi/2} + \xi e^{-\varphi/2}$$

as

$$\varphi \circ \omega = \omega \circ \varphi + 2\xi e^{-\varphi}.$$

By induction we obtain

$$e^{n\varphi} \circ \omega - \omega \circ e^{n\varphi} = 2n\xi e^{(n-1)\varphi}.$$

Using an analytic continuation (formally it is necessary to introduce $z = e^{\varphi}$) we come to conclusion that

$$e^{a\varphi} \circ \omega - \omega \circ e^{a\varphi} = 2a\xi e^{(a-1)\varphi}\,.$$

Using induction in m we get

$$\omega^m \circ e^{a\varphi} = \sum_{p=0}^{m} (-1)^p \binom{m}{p} a(a-1)\ldots(a-p+1)(2\xi)^p e^{(a-p)\varphi} \circ \omega^{m-p}\,. \tag{3}$$

Recall, that to find the twist it is necessary to compute the coefficient at 1 in the expression $(\varphi^n \circ \omega^p) \cdot (\varphi^m \circ \omega^q)$ when we reduce it to the form $\sum C_{rs}^{np,mq} \varphi^r \circ \omega^s$. In other words, it is necessary to find $C_{00}^{np,mq}$.

From the relation (3) it follows that if n and q are different from zero then $C_{00}^{qm,kn} = 0$. So, it is necessary to compute $C_{00}^{0m,k0}$. Moreover, it is obvious that it is enough to consider only last term, namely

$$(-1)^m (2\xi)^m a(a-1)\ldots(a-m+1)e^{(a-m)\varphi}\,.$$

To find the commutation relations for the elements ω^m and φ^k we must differentiate the relation (3) in "variable a " several times and then put $a = 0$. In particular, it is easy to see, that

$$C_{00}^{0m,k0} = \frac{(-2\xi)^m}{m!k!} \cdot \frac{d^k}{da^k}(a)_m \Big|_{a=0}\,,$$

where $(a)_m = a(a-1)\dots(a-m+1)$. Then

$$\mathcal{F} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}(2\xi)^m \cdot x^m \otimes \sum_{k=0}^{\infty} \frac{h^k}{k!}\frac{d^k}{da^k}(a)_m\Big|_{a=0}.$$

where h, x are generators of the algebra $U(b_2)$.

The second factor in the tensor product is the expansion of $(a)_m$ in the Taylor series in a neighborhood of the point $a = h$. Thus, finally it yields

$$\mathcal{F} = \sum_{m=0}^{\infty} \frac{1}{m!}(-2\xi x)^m \otimes (h)_m = (1\otimes 1 - 2\xi x \otimes 1)^{1\otimes h} = \exp(\ln(1-2\xi x)\otimes h)\,.$$

It is easy to check that in the fundamental representation the jordanian R-matrix has been reproduced

$$(\rho\otimes\rho)(\mathcal{F}_{21}\mathcal{F}^{-1}) = \begin{pmatrix} 1 & -\xi & \xi & \xi^2 \\ 0 & 1 & 0 & -\xi \\ 0 & 0 & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3.2 One more example of twist for $U(b_2)$

As another elementary example we will consider the following unitary quasiclassical solution of the Yang - Baxter equation (see, for example, [5]):

$$\boldsymbol{R}_{triang} = \begin{pmatrix} 1 & -\xi & \xi & -\xi^2 \\ 0 & 1 & 0 & \xi \\ 0 & 0 & 1 & -\xi \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 + \xi r + \mathcal{O}(\xi^2)\,, \tag{4}$$

where $r = \sigma^+\otimes\sigma^0 - \sigma^0\otimes\sigma^+$ and $\sigma^0 = \mathrm{diag}(1,1)$. From the structure of the classical r-matrix it follows that for the construction of a carrier for the r-matrix it is necessary to add the central element c to the Borel algebra from the previous example. As dual bases in U and U^* we will choose

$$h^m\, x^k\, c^p \quad \text{and} \quad \frac{\varphi^m\,\omega^k\,\alpha^p}{m!\,k!\,p!} \tag{5}$$

The canonical element in $U\otimes U^*$ is equal to

$$\mathcal{T} = \exp\left(h\otimes\varphi\right)\exp\left(x\otimes\omega\right)\exp\left(c\otimes\alpha\right), \tag{6}$$

and in the representation $\rho(c) = 1,\ \rho(h) = \frac{1}{2}\sigma^z,\ \rho(x) = \sigma^+$ we have

$$\mathcal{T} = e^{h\otimes\varphi}e^{x\otimes\omega}e^{c\otimes\alpha}\Big|_{\{\rho: c=1\,,\, h=\frac{1}{2}\sigma^z\,,\, x=\sigma^+\}} = \begin{pmatrix} e^{\varphi/2} & 0 \\ 0 & e^{-\varphi/2}\end{pmatrix}\begin{pmatrix} 1 & \omega \\ 0 & 1\end{pmatrix}\begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{\alpha}\end{pmatrix} \tag{7}$$

The relation $RTT = TT$ with the R-matrix (4) defines a new (noncommutative) multiplication law for the generators $\varphi,\ \omega,\ \alpha$ of U^*

$$\varphi\alpha = \alpha\varphi, \quad \varphi\omega = \omega\varphi, \quad \alpha\omega = \omega\alpha + \xi\exp(-\varphi). \tag{8}$$

To find the structure constants of the new multiplication law we take into account that the element φ is central and the commutation relations (8)

$$\varphi^m\omega^k\alpha^p \circ \varphi^{m_1}\omega^{k_1}\alpha^q = \varphi^{m+m_1}\omega^{k+k_1}\alpha^{p+q} + \dots . \tag{9}$$

To compute the twist $\mathcal{F}$ we must find nonzero structure constants $m_{(000)}^{(mkp)(m_1k_1q)}$. It is possible only if we put $m = m_1 = 0,\ k = q = 0,\ p = k_1$. Using (8) we we get for $(k \leq p)$:

$$\alpha^p\omega^k = \omega^k\alpha^p + \sum_{n=0}^{k}\omega^n\alpha^{p-k+n}C_k^n\xi^{k-n}C_p^{k-n}(k-n)!\, e^{-(k-n)\varphi} . \tag{10}$$

Thus, the structure constants (with $k = p, n = 0$) we are looking for are equal to

$$m_{(000)}^{(00p)(0p0)} = \frac{\xi^p}{p!},$$

and the corresponding universal twist has the following form

$$\mathcal{F} = \sum_p \frac{\xi^p}{p!}c^p \otimes x^p = \exp(\xi c \otimes x) . \tag{11}$$

In the vector representation $\mathbb{C}^2$ $\mathcal{F}$ is a 4×4-matrix and the expression $R = F_{21}F^{-1}$ coincides with the initial R-matrix.

3.3 The case of the Lie superalgebra sb_2

Let us consider the Lie superalgebra sb_2, which also has two generators: one even h and one odd x with the commutation relations

$$[h, x] = x, \quad [x, x] = 0 .$$

In the universal enveloping superalgebra $U(sb_2)$ the commutator $[,]$ is understood as a supercommutator ($\mathbb{Z}_2$-graded commutator):

$$[a, b] = ab - (-1)^{p(a)p(b)}ba .$$

We will choose as the unitary quasiclassical solution of the Z_2-graded Yang - Baxter equation [11] the following R-matrix

$$R = \begin{pmatrix} 1 & \eta & -\eta & 0 \\ 0 & 1 & 0 & -\eta \\ 0 & 0 & 1 & -\eta \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{12}$$

where η is an odd Grassmannian parameter: $\eta^2 = 0$.

The generators of the dual algebra $U^*(sb_2) : \{u, \omega : \langle u, h\rangle = 1,\ \langle \omega, x\rangle = 1\},\ \omega^2 = 0$ are mutually commute. The canonical element $\mathcal{T} \in U \otimes U^*$ is equal

$$\mathcal{T} = e^{h\otimes u}\, e^{x\otimes\omega}\,. \tag{13}$$

Similar to the first example the superalgebra sb_2 has the two-dimensional representation:

$$T = \begin{pmatrix} e^{u/2} & -e^{u/2}\omega \\ 0 & e^{-u/2} \end{pmatrix},$$

We fix the following dual bases in U and U^*

$$h^m x^k \quad \text{and} \quad u^m \omega^k / m!\,,$$

where $(m = 0, 1, \cdots ; k = 0, 1)$. The relation $RTT = TT$ with R-matrix (12) defines a new (noncommutative) multiplication law for the generators u, ω of the dual superalgebra U^*

$$\omega u = u\omega - 2\eta e^{-u}, \quad \omega^2 = 0. \tag{14}$$

To define the structure constants of this new multiplication law we take into account commutation relations (14). In order to calculate $\mathcal{F}$ we need to find the structure constants $m_{(00)}^{(mk)(m_1k_1)}$ different from zero. For this it is necessary to set $m = 0,\ k_1 = 0,\ m_1 = k = 1$. As a consequence of (14), we obtain $m_{(00)}^{(01)(10)} = -2\eta$, and the corresponding universal twist takes the form

$$\mathcal{F} = 1 - 2\eta x \otimes h. \tag{15}$$

The universal R-matrix

$$\mathcal{R} = 1 - 2\eta(x \otimes h - h \otimes x) = \exp(-2\eta(x \otimes h - h \otimes x)) \tag{16}$$

is linear in the generator x due to nilpotency of x and η.

3.4 Twisting of quantum superalgebra $U_q(gl(1|1))$

The quantum superalgebra $U_q(gl(1|1))$ (see Appendix) is an example of the case with more complicated R-matrix. In the fundamental two-dimensional representation of the quantum superalgebra $U_q(gl(1|1))$ this R-matrix is equal to

$$R^{(t)} = \begin{pmatrix} q & 0 & 0 & \xi \\ 0 & q & \omega & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \tag{17}$$

The relation $R^{(t)}TT = TTR$ with the $R^{(t)}$-matrix (17) defines a new (noncommutative) multiplication law for the elements (generators) of $U_q(gl(1|1))^* : \alpha, \varphi, b, g$ entering the T-matrix. To

write down explicit forms of matrices T_1 and T_2 it is necessary to include signs related to the Z_2-grading (see Appendix):

$$T_1 = T \otimes I = \begin{pmatrix} a & 0 & u & 0 \\ 0 & a & 0 & -u \\ w & 0 & d & 0 \\ 0 & -w & 0 & d \end{pmatrix}, \quad T_2 = I \otimes T = \begin{pmatrix} a & u & 0 & 0 \\ w & d & 0 & 0 \\ 0 & 0 & a & u \\ 0 & 0 & w & d \end{pmatrix}. \tag{18}$$

In the right hand side of the $RTT = TT$-relation we use the R-matrix with $\xi = 0$. The only commutation relation, giving nonzero structure constant $m^{(j)(k)}_{(0)}$ with nonzero multi-indices needed to calculate the twist $\mathcal{F}$ is

$$u^2 = \frac{\xi}{q+q^{-1}} d^2, \quad \text{or} \quad b^2 = \frac{\xi}{q+q^{-1}} 1 + \cdots ,$$

where the terms containing generators different from unit are omitted. Thus, $m^{(0001)(0001)}_{(0000)} = -\xi/(q+q^{-1})$ and the twist element is

$$\mathcal{F} = \exp(-\frac{\xi}{q+q^{-1}} e \otimes e) = 1 - \frac{\xi}{q+q^{-1}} e \otimes e \, .$$

4 Discussion

In a similar way it is possible to consider the quantum group $U_q(g\ell_n)$. It is known, that the dual Hopf algebra $Fun_q(GL_n)$ which is the algebra of functions on the quantum group, can be obtained from the relation $R_q T_1 T_2 = T_2 T_1 R_q$, where R_q is the Drinfel'd - Jimbo R-matrix . Then the algebra, defined by $R'_q T_1 T_2 = T_2 T_1 R_q$, will be a right module algebra over $U_q(g\ell_n)_{op}$. Here R'_q is another R-matrix such that $R'_q \mathcal{P}$ and $\mathcal{P} R_q$ have the same spectrum ($\mathcal{P}$ is the permutation operator in $\mathbb{C}^n \otimes \mathbb{C}^n$). Then there exists a universal twist $\mathcal{F} \in U_q(g\ell_n)^{\otimes 2}$, such that $R'_q = (\rho \otimes \rho)(\mathcal{F}_{21} \mathcal{R}_q \mathcal{F}^{-1})$, where $\mathcal{R}_q$ is the universal R-matrix and $\rho : U_q(g\ell_n) \to End(\mathbb{C}^n)$ is fundamental vector representation of the Hopf algebra $U_q(g\ell_n)$.

Appendix

A. Universal R-matrix for $gl_q(1|1)$

The superalgebra $U_q(gl(1|1))$ is generated by two even h, c, and two odd e, f elements where c is central, and the remaining generators satisfy the relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad e^2 = f^2 = 0, \quad [e, f]_+ = (q^c - q^{-c})/(q - q^{-1})$$

The Z_2-graded structure of Hopf algebra is defined by the following coproduct

$$\begin{aligned} \Delta(h) &= h \otimes 1 + 1 \otimes h, \\ \Delta(c) &= c \otimes 1 + 1 \otimes c, \\ \Delta(e) &= e \otimes q^c + 1 \otimes e, \\ \Delta(f) &= f \otimes 1 + q^{-c} \otimes f, \end{aligned}$$

and counit $\varepsilon(h) = \varepsilon(c) = \varepsilon(e) = \varepsilon(f) = 0$. The algebra $U_q(gl(1|1))$ is quasitriangular with the universal R-matrix

$$\mathcal{R} = q^{\frac{1}{2}(c\otimes h + h\otimes c)}(1 - \omega\, e\otimes f) \in U_q(gl(1|1))^{\otimes 2},$$

where $\omega = (q - q^{-1})$. This universal R-matrix intertwines the coproduct with the opposite one.

Let us remark that the superalgebra $gl_q(1|1)$ admits an abelian twist because it has a two-dimensional classical abelian subalgebra spanned by the elements c and h with primitive coproduct. There is a special choice of such a twist element that one of the odd generators, say e, becomes a primitive element. Indeed

$$\Delta_t(e) = e^{\alpha h\otimes c}\Delta(e)e^{-\alpha h\otimes c} = e\otimes q^c e^{2\alpha c} + 1\otimes e = e\otimes 1 + 1\otimes e$$

if we set $\alpha = -\frac{1}{2}\log(q)$.

Hence, there exists another twist $\mathcal{T}_1 = \exp(\xi e\otimes e)$. As a result of twisting by this element we obtain a new universal R-matrix

$$\widetilde{\mathcal{R}} = \exp(-\xi e\otimes e)e^{\alpha c\otimes h}\mathcal{R}e^{-\alpha h\otimes c}\exp(-\xi e\otimes e)\,,$$

having in the matrix representation an off-diagonal parameter.

B. Dual Hopf algebra $U_q(gl(1|1))^*$

Generators of the coquasitriangular Hopf algebra dual to $U_q(gl(1|1))$ are defined by nonzero pairings

$$\langle\varphi, h\rangle = 1, \quad \langle\alpha, c\rangle = 1, \quad \langle b, e\rangle = 1, \quad \langle g, f\rangle = 1.$$

The generators φ and α are even, whereas b, g are odd. The canonical element

$$\mathcal{T} \in U_q(gl(1|1))\otimes U_q(gl(1|1))^*$$

is defined by the standard exponentials due to nilpotency of the generators e, f (and b, g)

$$\mathcal{T} = \exp(f\otimes g)\exp(c\otimes\alpha)\exp(h\otimes\varphi)\exp(e\otimes b),$$

$$\mathcal{T} = (1 + f\otimes g)\exp(c\otimes\alpha)\exp(h\otimes\varphi)(1 + e\otimes b).$$

The duality relations allows us to calculate coproduct and to find the multiplication law for the generators of the dual Hopf algebra

$$\begin{aligned}
\Delta_*(\varphi) &= \varphi\otimes 1 + 1\otimes\varphi,\\
\Delta_*(\alpha) &= \alpha\otimes 1 + 1\otimes\alpha - \frac{2\eta}{\omega}b\otimes g,\\
\Delta_*(b) &= b\otimes\exp(-2\varphi) + 1\otimes b,\\
\Delta_*(g) &= g\otimes 1 + \exp(-2\varphi)\otimes g,
\end{aligned}$$

$$[\alpha, b] = -\eta\, b, \quad [\alpha, g] = -\eta\; g, \quad b^2 = g^2 = 0, \quad [b, g]_+ = 0.$$

To check that the canonical element $\mathcal{T}$ is a bicharacter

$$(\Delta\otimes 1)\mathcal{T} = \mathcal{T}_{13}\mathcal{T}_{23}\,, \quad (1\otimes\Delta_*)\mathcal{T} = \mathcal{T}_{12}\mathcal{T}_{13}\,,$$

it is necessary (using the coproduct Δ in the Hopf algebra $U_q(gl(1|1))$ and dual coproduct Δ_* in $U_q(gl(1|1))^*$) to take into account a particular form of the Baker - Campbell - Hausdorff formula

$$\exp(tX + Y) = \exp(tX)\exp(\frac{1-e^{-t}}{t}Y), \quad [X,Y] = Y.$$

This formula allows us to extract a factor in $(1 \otimes \Delta_*)\mathcal{T}$, namely,

$$\exp(c \otimes (\alpha \otimes 1 + 1 \otimes \alpha - \frac{2\eta}{\omega} b \otimes g)) =$$
$$= \exp(c \otimes \alpha \otimes 1)\exp(-\frac{(q^c - q^{-c})}{\omega} b \otimes g))\exp(c \otimes 1 \otimes \alpha)\,,$$

due to the commutation relation

$$[c \otimes \alpha \otimes 1 + c \otimes 1 \otimes \alpha, 1 \otimes b \otimes g] = -2\eta c \otimes b \otimes g.$$

In the two-dimensional representation of the Hopf super-algebra $U_q(gl(1|1))$ the elements of the 2×2-matrix $T = \begin{pmatrix} a & u \\ w & d \end{pmatrix}$ can be expressed in terms of the generators of the dual superalgebra $U_q(gl(1|1))^*$

$$a = e^{(\alpha+\varphi)}\,, \quad u = e^{(\alpha+\varphi)}b, \quad w = ge^{(\alpha+\varphi)}\,, \quad d = e^{(\alpha-\varphi)} + ge^{(\alpha+\varphi)}b\,.$$

References

[1] V. Drinfel'd, *Quantum groups.* In: *Proceedings of the International Congress of Mathematician*, Berkeley, CA, 1986. AMS, Providence, RI, 1987, p. 798.

[2] V. Drinfel'd, *Constant semiclassical solutions of the quantum Yang - Baxter equation*, DAN USSR **273**, no.3 (1983) 531-535 (in Russian).

[3] V. Drinfel'd, *Quasi-Hopfian algebras*, Algebra and Analysis **1:6** (1989) 114-148 (in Russian).

[4] N.Yu. Reshetikhin, L.A. Takhtajan, L.D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra and Analysis, **1**, no.1 (1989) 178-206 (in Russian).

[5] P.P. Kulish, V.D. Lyakhovsky, A.I. Mudrov, *Extended jordanian twists for Lie algebras*, J.Math. Phys. **40**, no.9 (1999) 4569-4581.

[6] V.D. Lyakhovsky, A.A. Stolin, P.P. Kulish, *Chains of Frobenius subalgebras of* $so(M)$ *and the corresponding twists*, J. Math. Phys. **42**, no.10 (2001) 5006-5019.

[7] S. Parmentier, *On coproducts of quasitriangular Hopf algebras*, Algebra and Analysis **6**, .4 (1994) 204-222.

[8] S. Zakrzewski, *A characterization of coboundary Poisson Lie groups and Hopf algebras*, Banach Center Publ. **40** (1997) 273-278.

[9] E.E. Demidov, Yu. I. Manin, E.E. Mukhin, D.V. Zhdanovich, *Non-standard quantum deformations of $GL(n)$ and constant solutions of the Yang-Baxter equation*, In: Yukawa seminar *Common Trends in Mathematics and Quantum Field Theories*. Ed. by T.Eguchi, T.Inami and T.Miwa. Prog. Theor. Phys. Suppl. **102** (1990) 203-218.

[10] P.P. Kulish, A.I. Mudrov, *Quantization of inhomogeneous Lie bialgebras*, J.Geom. Phys. **42** (2002) 64 - 77.

[11] P.P. Kulish, E.K. Sklyanin, *Quantum spectral transform method: recent developments*, In: *Integrable Quantum Field Theories*. Ed. by J.Hietarinta, C.Montonen; *Lect. Notes Phys.*, **151** (1982) 61-119.

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